THE OSCILLATIONS OF YVES MEYER

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Abstract. In this review, some of Yves Meyer's contributions to the analysis of pointwise exponents and fluctuations of local regularity are gathered. An emphasis is given to the description of oscillatory singularities and their important role in different fields of Mathematics, as well as in some applications to other scientific fields.

1. Introduction

Yves Meyer's personality and results undoubtedly influence the career of many researchers. In fact, he played an important role in my journey leading me to become a mathematician. In 1998, he first introduced me to Ingrid Daubechies which allowed me to spend four months in the Fine Hall building of Princeton University where I discovered the beautiful theory of wavelets. Then, between 2001 and 2004, during regular one-to-one meetings, I exposed to Yves the results of my on-going PhD thesis, and benefited from his insightful advice regarding my proofs and also what it takes to be a mathematician.

One piece of advice particularly struck me: Yves explained how important it is to switch subjects every ten years to maintain one's appetite, curiosity and interest for Mathematics. And indeed, although having become each time one of the leaders in the field, he went from quasi-crystals to harmonic analysis and the Calderon-Zygmund program, then to wavelet theory, to PDEs, and finally to the mathematical aspects of the signal and the image processing (awaiting the next subject). Although I am convinced by his philosophy and mind set, and more importantly his way of applying it successfully, I believe that this is a path that unfortunately only a few mathematicians, the exceptional ones, can take. [1]

Yves's ability to listen to problems coming from other communities, combined with his extraordinary talent for discovering hidden connections while connecting dots between different areas, is probably one of the reasons he is involved in oscillatory behaviors and multifractal questions for nearly thirty years. His interest probably originated in the 1980s in the seminal conferences around wavelets, during which researchers from harmonic and Fourier analysis, signal processing and Physics, interacted very enthusiastically, beyond usual boundaries between scientific fields. There were many challenges (some of them still open) around quantifying and measuring fluctuations of pointwise regularity and local oscillations for many signals, and in particular over detecting "fast" oscillating singularities in signals. These issues have been raised, for example, by Alain

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[1]The title of this article refers to the comic "Les oscillations de Joseph Fourier" [17], in which the various lives (as scientific advisor in the Egyptian campaign of Napoléon, scientist, prefect, ..) of Joseph Fourier are marvelously recounted.

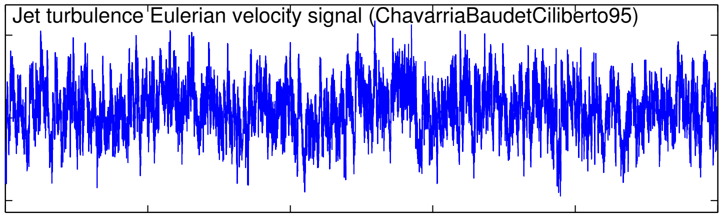

FIGURE 1. Observed 1d-velocity of a turbulent fluid.

Arneodo and his team and P. Flandrin (both at ENS Lyon), A. Grossmann and B. Torresani in Marseille, as well as by many scientists working on turbulence (U. Frisch and G. Parisi) and physicists enthusiastic about the prospect of detecting gravitational waves. In many phenomena, the presence of "chirps", i.e. points around which functions/signals/data have a rapid oscillatory behavior, was a key to validate or refute some theories. Oscillatory behaviors were also pointed out using wavelet tools in [24] by Holschneider and Tchamitchian for the Riemann Fourier series

$$R(x) = \sum_{n\geq 1} \frac{\sin(2\pi n^2 x)}{n^2}, \tag{1}$$

bringing a new outlook on the reasons why R is differentiable (only) at some rational points. The wavelet methods developed by Morlet, Grossmann, Mallat, Meyer, Daubechies, and Jaffard, opened up new perspectives in Mathematics and signal analysis, and Yves was naturally enrolled in this further understanding of local regularity and oscillatory behavior. These works had many consequences in Mathematics and in Physics, recently culminating in the proof of the existence of gravitational waves by the detection of chirps using wavelet tools [1, 18].

The purpose of this article is to describe some of Yves Meyer's contributions to the field of pointwise analysis of functions and distributions, and of multifractal analysis. Although these are not the fields in which he has contributed the most, his regular publications over the past thirty years and his enthusiastic participation in each session of the Séminaire Cristolien d'Analyse Multifractale attest to his constant interest in these subjects. We first recall the first results of [32] on the wavelet characterization of local modulus of continuity. Then, based on [39], we explain how Yves Meyer used 2-microlocal analysis and the 2-microlocal space $C^{s,s'}_{x_0}$ to provide a very complete description of the pointwise behavior of a function or a distribution. Further, the decomposition of functions belonging to $C^{s,s'}_{x_0}$ as a sum of a regular term and an oscillatory term is given, justifying the introduction to these spaces. In particular, this allows to detect points x_0 which are *chirps* for a real function $f : \mathbb{R}^d \to \mathbb{R}$, i.e. points around which, up to a smooth term, f locally behaves like

$$x \mapsto |x|^\alpha g(|x|^{-\beta}),$$

where g is an indefinitely oscillating function (typically, a sine or cosine). This was the ultimate goal in many fields, since the existence of such decompositions is key to understand and model some of the phenomena discussed before. This is applied in particular to the Riemann Fourier series (see (1) below). Finally, we give some heuristics and results about the reasons why the presence of chirps make the so-called multifractal formalism fails in Section 6.

2. A short reminder on Littlewood-Paley analysis and wavelets

2.1. The Littlewood-Paley decomposition of a tempered distribution. Let $\phi \in \mathcal{S}(\mathbb{R}^d)$ be a function in the Schwarz class satisfying

$$\widehat{\phi}(\xi) = 1 \text{ if } |\xi| \leq 1/2 \quad \text{and} \quad \widehat{\phi}(\xi) = 0 \text{ if } |\xi| \geq 1, \tag{2}$$

where $\widehat{\phi}$ is the Fourier transform of ϕ. This function ϕ must be understood as a low-pass filter. Then, one defines, for every integer $j \in \mathbb{Z}$, the operator S_j on the space of tempered distribution $\mathcal{S}'(\mathbb{R}^d)$ by its action in the Fourier domain as

$$\widehat{S_j f}(\xi) = \widehat{f}(\xi)\widehat{\phi}(2^{-j}\xi).$$

The operator $\Delta_j = S_{j+1} - S_j$ is naturally associated with S_j, and is interpreted as a band-pass filter since $\Delta_j f$ vanishes outside the annulus $2^{j-1} \leq |\xi| \leq 2^{j+1}$. Using that

$$I_d = S_0 + \sum_{j=0}^{+\infty} \Delta_j,$$

the family $(S_0 f, (\Delta_j f)_{j\geq 0})$ is called the Littlewood-Paley decomposition of f.

The Littlewood-Paley decomposition carries important information regarding the regularity properties of f. For instance, the decay rate of $\|\Delta_j f\|$ is related to the global Hölder regularity of f, see next section.

This decomposition is not completely satisfactory when one is interested in local behaviors, since, for instance, $|\Delta_j f|$ is not easily accessible by numerical estimates. Wavelets, that can be thought of as a discretization of the Littlewood-Paley decomposition, are more suited to the study of pointwise behavior.

2.2. Continuous wavelet transform.

Definition 1. Let $\psi \in \mathcal{S}(\mathbb{R}^d)$ be a radial function, having a vanishing integral, i.e. $\displaystyle\int_{\mathbb{R}^d} \psi(u)du = 0$.

The continuous wavelet transform associated with ψ of a function $f \in L^2(\mathbb{R}^d)$ is defined for every $(a, b) \in \mathbb{R}_+^* \times \mathbb{R}^d$ by

$$W_f(a,b) = a^{-d/2} \int_{\mathbb{R}} f(x)\psi_{a,b}(x)dx \quad \text{where} \quad \psi_{a,b}(x) = \psi\left(\frac{x-b}{a}\right). \tag{3}$$

The decay rate to zero of $|W_f(a,b)|$ when a tends to 0 depends on the regularity properties of f. But this time, a localization parameter b is added. Since ψ is localized around 0, $\psi_{a,b}$ is localized around b, and the analysis of the coefficients $W_f(a,b)$ for b close to $x_0 \in \mathbb{R}^d$ reflects the local behavior of f around x_0.

2.3. **Discrete wavelet transform.** Let ϕ be a scaling function and $\{\psi^{(i)}\}_{i=1,\dots,2^d-1}$ be a family of wavelets associated with ϕ such that $(\phi, \{\psi^{(i)}\}_{i=1,\dots,2^d-1})$ defines a multi-resolution analysis with reconstruction in $L^2(\mathbb{R}^d)$ (see [37] for a general construction). We refer the reader to other chapters in the present book in which Meyer's contributions to wavelet theory are detailed.

For every $(i,j,k) \in \Lambda := \{1,...,2^d-1\}\times\mathbb{Z}\times\mathbb{Z}^d$, denote by $\psi_\lambda(x) := 2^{jd/2}\psi^{(i)}(2^jx-k)$. Then, the family $\{\psi_\lambda : \lambda \in \Lambda\}$ forms an orthonormal basis of $L^2(\mathbb{R}^d)$, so that every $f \in L^2(\mathbb{R}^d)$ can be expanded as

$$f = \sum_{j\in\mathbb{Z}} \sum_{\lambda\in\Lambda_j} d_\lambda \psi_\lambda, \quad \text{with} \quad d_\lambda = \langle f, \psi_\lambda \rangle = \int_{\mathbb{R}^d} \psi_\lambda(x) f(x)\,\mathrm{d}x. \tag{4}$$

Again, it follows from their construction that the wavelets $\psi^{(i)}$ are localized in time and in frequency. So, ψ_λ is localized in time around the dyadic point $k2^{-j}$ and in frequency around 2^j. From this, given $x_0 \in \mathbb{R}^d$, one deduces that the local behavior of f around x_0 influences the value of the *wavelet coefficients* d_λ when $\lambda = (i,j,k)$ is such that $k2^{-j}$ is close to x_0.

Wavelets are oscillating functions, that are characterized by their smoothness and their number of vanishing moments: ψ has N vanishing moments when for every $\alpha = (\alpha_1,...,\alpha_d) \in N^d$ with $|\alpha| := \alpha_1 + \cdots + \alpha_d \le N$, $\int_{\mathbb{R}^d} \psi(x) x^\alpha dx = 0$, where for $x = (x_1,...,x_d) \in \mathbb{R}^d$ and, $x^\alpha := x_1^{\alpha_1} \cdots x_d^{\alpha_d}$. The larger N, the more ψ oscillates, and wavelet coefficients computed with wavelets having N vanishing moments can "catch" Hölder regularity exponents less than N (see Theorem 1 below).

The existence of compactly supported wavelets with arbitrary number of vanishing moments and arbitrary global Hölder regularity is due to I. Daubechies [20]. These wavelets are key for applications, since using compact wavelets makes the numerical estimates much faster.

3. Characterization of the pointwise behavior of functions

As explained in the introduction, in many phenomena (especially in Physics), it is common to observe very irregular data. More precisely, on one hand many of these data apparently possess some global Hölder regularity, but on the other hand, they vary in an erratic way: not only do they seem not to be differentiable, but their pointwise regularity changes from time to time. For instance, the intensity of the 1D-velocity of a turbulent fluid was proved to have these fluctuations properties (see Figure 1), and led U. Frisch and G. Parisi to the notion of multifractal formalism in [22]. This is also the case for physiological data such as EEG and ECG signals, see Figure 3. This was observed earlier by Mandelbrot in many geophysical and meteorological data, in finance, texture analysis,.. Even more, in some physical models, the fast variation of pointwise regularity may indicate the occurence of a singular event (for instance, the collapse between a binary pair of black holes), and the detection of such singularities and the way they are distributed in time and/or space is a key issue.

These pointwise regularity fluctuations also arise for various families of mathematical objects: the famous Riemann's lacunary Fourier series (1) is known to be not differentiable at all $x \in \mathbb{R}$ but the rational numbers $\frac{2p+1}{2q+1}$, $p, q \in \mathbb{Z}$ (see [23, 25, 28]). Typical

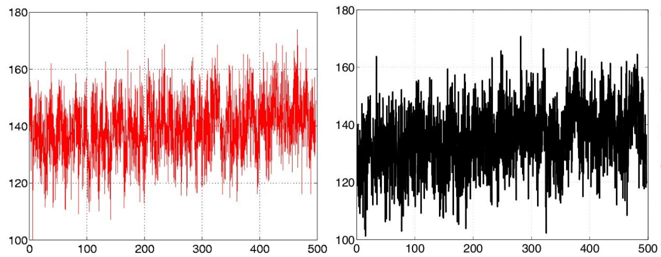

FIGURE 2. EEG signals. Credits: P. Abry (ENS Lyon), H. Wendt (IRIT, Toulouse).

functions in many function spaces (spaces of increasing functions [13] or measures [14], Besov spaces [31], see also [8, 9]) have a varying pointwise regularity, and many of them also possess chirp-like singularities (these objects are mentioned here since Yves Meyer contributed directly or indirectly to their analysis). Such results go against the intuition that a "typical" function has the same pointwise behavior at every point or is C^∞. Although we do not elaborate on this here, the famous Mandelbrot cascades, Gibbs measures invariant by the shift on the torus, and many other measures, also have generically a varying pointwise regularity (measured by their local dimension).

Let us recall first the definition of the most common index used to describe the pointwise behavior of locally bounded functions.

Definition 2. Let $d \geq 1$ be an integer. Given $H \geq 0$, $f \in L^\infty_{loc}(\mathbb{R}^d)$ and $x_0 \in \mathbb{R}^d$, f is said to belong to $C^H(x_0)$ when there exists a polynomial P of degree less or equal than H and a constant $C > 0$ such that

$$\text{for } x \text{ close to } x_0, \quad |f(x) - P(x - x_0)| \leq C|x - x_0|^H. \tag{5}$$

The characterization of the space $C^H(x_0)$ by wavelet analysis, although classical now, is not an immediate issue. It is immediate that $C^H(x_0) \subset C^{H'}(x_0)$ for $H' \leq H$, leading to the notion of pointwise Hölder exponent.

Definition 3. The *pointwise Hölder exponent* of $f \in L^\infty_{loc}(\mathbb{R}^d)$ at x_0 is

$$h_f(x_0) = \sup \left\{ H \geq 0 : \ f \in C^H(x_0) \right\},$$

and f is said to have a Hölder singularity of order $h_f(x_0)$ at x_0.

This index reflects indeed the pointwise behavior of the function under study. The characterization of the space $C^\alpha(x_0)$ by wavelet coefficients has been studied by S. Jaffard in the early 1990's [26].

One may be interested in finer results than only a power-law decay like (5). A standard example is provided by the study of fast and slow points of the Brownian motion initiated by Orey and Taylor [42, 43] and Kahane [34, 35], who proved that

$$\max_{t\in[0,1]} \left\{ \limsup_{h\to 0^+} \frac{|B(t+h) - B(t)|}{\sqrt{2h\log(1/h)}} \right\} = 1,$$

and that

$$\dim \left\{ t \in [0,1] : \limsup_{h\to 0^+} \frac{|B(t+h) - B(t)|}{\sqrt{2h\log(1/h)}} \geq \alpha \right\} = 1 - \alpha^2.$$

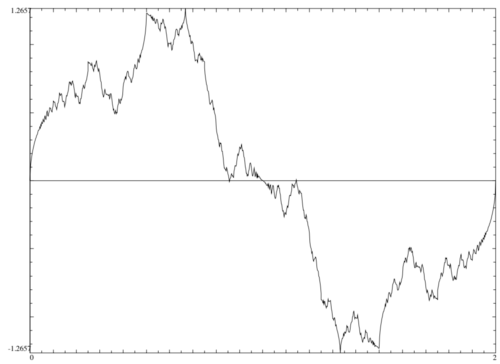

FIGURE 3. The Riemann lacunary Fourier series (1)

This example shows how important it is to develop both theoretical and practical tools able to detect such local fluctuations, which are detected using modulii of smoothness.

Definition 4. An increasing mapping $\theta : \mathbb{R}^+ \to \mathbb{R}^+$ is a modulus of continuity when
(i) $\theta(0) = 0$,
(ii) There exists $C > 0$ such that for every $x \geq 0$, $\theta(2x) \leq C\theta(x)$.

As explained above, finding precise modulii of smoothness is an important issue in many domains.

Definition 5. A function $f \in L^\infty_{loc}(\mathbb{R}^d)$ has θ as *uniform modulus of continuity* when there exists $C > 0$ such that

$$\forall h \in \mathbb{R}_+, \ w_f(h) := \sup_{|x-y|\leq h} |f(x) - f(y)| \leq C\theta(h).$$

A function $f \in L^\infty_{loc}(\mathbb{R}^d)$ has θ as *local modulus of continuity* at $x_0 \in \mathbb{R}^d$ when there exist $\delta, C > 0$ such that for every x such that $|x - x_0| \leq \delta$,

$$|f(x) - f(x_0)| \leq C\theta(|x - x_0|). \tag{6}$$

In [33], S. Jaffard and Y. Meyer extended the characterization previously obtained by Jaffard in [26] to adapt them to the description of general modulii of smoothness, and proved the following theorem.

Theorem 1. *Let $f \in L^\infty_{loc}(\mathbb{R}^d)$, and $x_0 \in \mathbb{R}^d$.*

- *If θ is a modulus of continuity for f at x_0, then there exists $C > 0$ such that for all x close enough to x_0,*

$$|\Delta_j f(x)| \leq C(\theta(2^{-j}) + \theta(|x - x_0|)). \tag{7}$$

- *If (7) holds and in addition there exists $\alpha > 0$ such that $h \mapsto |h|^\alpha$ is a uniform modulus of continuity for f, then there exist a polynomial P and a constant $C > 0$ such that locally around x_0*

$$|f(x) - P(x - x_0)| \leq C\theta(|x - x_0|)|\log\theta(|x - x_0|)|^{1+\alpha}. \tag{8}$$

In [32], Theorem 1 is proved provided a finer assumption than $h \mapsto |h|^\alpha$ in the converse part of the theorem.

There is an equivalent result with continuous and discrete wavelets, where (7) is replaced respectively by

$$|W_f(a,b)| \leq C(\theta(a) + \theta(|b - x_0|)) \text{ and } |d_\lambda| \leq C(\theta(2^{-j}) + \theta(|k2^{-j} - x_0|).) \tag{9}$$

and (8) remains the same.

Such characterizations are called for in many contexts. They were determinant to analyze the non-differentiability of historical mathematical objects, like the Riemann Fourier series (1). They are also used to study the pointwise regularity (and actually, the multifractality) of typical functions in critical Sobolev [33] and Besov spaces [31], of traces of functions [5], self-similar functions [11] ... Equations (7) and (9) are especially useful in random contexts, for instance they are key to analyze the pointwise regularity of random wavelet series [4], random sums of functions [44], sample paths of stochastic processes such as Lévy or Markov processes or solution to stochastic differential equations [30, 7, 49, 48].

Unfortunately, the sole pointwise Hölder exponent and the modulii of continuity do not describe completely the pointwise behavior of a function, and are not adapted to distributions or even to non-locally bounded functions.

In particular, they do not catch the local oscillations: it is trivial to check that for $\alpha > 0$, $x \mapsto |x|^\alpha$ and $x \mapsto |x|^\alpha \sin(|x|^{-\beta})$ share the pointwise Hölder exponent α and modulus of continuity at 0, but they obviously behave differently at 0. The first type of singularity is referred to as *cusps*, while the second, oscillating, one is usually called a *chirp*.

Also, another drawback of the pointwise Hölder exponent is that it is not stable under the action of pseudo-differential operators (for instance $f \in C^\alpha(x_0) \cap C^1(\mathbb{R})$ does not imply that $f' \in C^{\alpha+1}(x_0)$), this being essentially due to possible oscillatory behaviors. For instance,

Let us finally mention that some algorithms have been proposed to estimate $h_f(x_0)$, the most famous one being the wavelet transform modulus maxima (WTMM) method proposed by A. Arnéodo [3], S. Mallat [36] and their collaborators. We will come back to this in a few lines.

Many exponents, complementary to $h_f(x_0)$, have been introduced to circumvent this lack of completeness:

- the local Hölder exponent $\alpha_f(x_0) = \lim_{\varepsilon\to 0} \inf\{\beta > 0 : f \in C^\beta(B(x_0,\varepsilon))\}$, which describes the best global Hölder regularity of f on small neighborhoods of x_0,
- the weak scaling exponent $\tilde{\beta}_f(x_0)$ introduced in [39], which is the smallest real number s satisfying that $\tilde{\beta}_f(x_0) = s$ is equivalent to $\tilde{\beta}_{\partial_j f(x_0)} = s - 1$ (for every partial derivative ∂_j). This weak scaling exponent is proved by Y. Meyer to play an important role, since this exponent reflects what happens in the "cone of influence" of the wavelet transform at x_0, i.e. $\lambda = (i,j,k) : |x_0 - k2^{-j}| \leq 2^{-j}$. This exponent is actually related to the parameters estimated by the "Marseille Algorithm" developed by Grossmann and Torresani and the WTMM algorithm mentioned above [39].
- the oscillation exponent, defined as

$$\beta_f(x_0) = \lim_{s\to 0} \frac{\partial}{\partial s}\left(h_{f^{(-s)}}(x_0)\right) - 1, \tag{10}$$

where $f^{(-s)} = (Id - \Delta)^{-\frac{s}{2}}(\phi \circ f)$ is the s-fractional primitive of f.

The heuristics leading to $\beta_f(x_0)$ is the following: any primitive $F_{\alpha.\beta}$ of the chirp $f_{\alpha,\beta}(x) = |x|^\alpha \sin(|x|^\beta)$ has a pointwise Hölder exponent at 0 equal to $\alpha + \beta + 1$, hence $\beta = h_{F_{\alpha.\beta}}(0) - h_{f_{\alpha,\beta}}(0) - 1$. Equation (10) is nothing but the rigorous formulation of this intuition at the infinitesimal level.

It is commonly said that a *cusp* is a point at which the singularity exponent equals 0, while a *chirp* is a point at which the singularity exponent is strictly positive.

Remark 1. This inverse problem consisting in finding functions with prescribed pointwise regularity has been addressed by Y. Meyer in [19] for instance, and also in [27] or [16, 6] for measures. It is indeed a very natural question for modeling purposes to investigate such questions.

In [39], Y. Meyer proposes a global approach to handle all these exponents (and more) simultaneously.

4. A finer characterization of local behaviors via 2-microlocal spaces

The problem is thus threefold:

(i) how to characterize using wavelets the local oscillations of continuous functions, to distinguish cusps from chirps,
(ii) how to define pointwise regularity exponents for non-locally bounded functions, typically L^2 functions, and distributions.
(iii) to find conditions under which a function can be locally written as $r(x - x_0) + |x - x_0|^\alpha g(|x - x_0|^{-\beta})$, where r is smooth and g is oscillating.

As explained by Meyer in [38] and [39], the 2-microlocal spaces, introduced by J.-M. Bony in [12], provide a quite satisfactory and elegant answer to the three questions.

Definition 6. Let $x_0 \in \mathbb{R}^d$, and $s, s' \in \mathbb{R}$. A tempered distribution f belongs to $C^{s,s'}_{x_0}$ when there exists a constant $C > 0$ such that for every $x \in \mathbb{R}^d$ close to x_0,

$$|S_0 f(x)| \leq C(1 + |x - x_0|)^{-s'}$$
$$|\Delta_j f(x)| \leq C 2^{-js}(1 + |x - x_0|)^{-s'}.$$

It can be proved that $C^{s,s'}_{x_0}$ is independent of the choice of ϕ in (2) used to define the Littlewood-Paley analysis of Definition 6.

In this article, we only consider local versions of these spaces, in the sense that $f \in C^{s,s'}_{x_0}$ means that there exists a function $g \in C^\infty$ with $g \equiv 1$ on a neighborhood of 0 and $g \equiv 0$ on $\{x : |x| \geq 1\}$ such that fg satisfies the properties of Definition 6.

It is known that $C^{s,s'}_{x_0}$ behaves nicely under the action of differential operators: for instance,

$$f \in C^{s,s'}_{x_0} \Leftrightarrow (\text{for any } \alpha = (\alpha_1, ..., \alpha_d) \in N^d,\ \partial^\alpha f \in C^{s-|\alpha|,s'}_{x_0}) \tag{11}$$

Meyer's idea is then to describe the local regularity not by a single exponent, or a collection of exponents, but by a whole region in the (s, s')-plane, defined by the indices (s, s') such that f belongs to $C^{s,s'}_{x_0}$.

Definition 7. Let $x_0 \in \mathbb{R}^d$, and $f \in \mathcal{S}'(R^d)$. The 2-microlocal domain of f at x_0 is the set $E_f(x_0) = \{(s, s') \in \mathbb{R}^2 : f \in C^{s,s'}_{x_0}\}$.

It can be shown from the embedding properties of the spaces $C^{s,s'}_{x_0}$ that the border $\Gamma_f(x_0)$ of the 2-microlocal domain $E_f(x_0)$, viewed as a mapping $\mathbb{R} \to \mathbb{R} \cup \{-\infty, +\infty\}$ in the (s, s')-plane, is a concave, decreasing function with left and right derivatives smaller than -1. For a given locally bounded function, the knowledge of $E_f(x_0)$ contains much more information than the pointwise Hölder exponent, as stated by the following proposition.

Proposition 2. *Let $f \in L^\infty_{loc}(\mathbb{R}^d)$.*

- *When $f \in C^\varepsilon(\mathbb{R}^d)$ for some $\varepsilon > 0$, the pointwise Hölder exponent is*
$$h_f(x_0) = \sup\{s \geq 0 : f \in C^{s,-s}_{x_0}\} = \sup\{s : (s, -s) \in \Gamma_f(x_0)\}.$$
- *The local Hölder exponent is*
$$\alpha_f(x_0) = \sup\{s : f \in C^{s,0}_{x_0}\} = \sup\{s \geq 0 : (s, 0) \in E_f(x_0)\}.$$
- *The weak scaling exponent $\beta^w_f(x_0)$ introduced in* [39] *is*
$$\beta^w_f(x_0) = \lim_{n\to-\infty} \sup\{s : f \in C^{s,n}_{x_0}\} = \lim_{n\to-\infty} \sup\{s : (s, n) \in E_f(x_0)\}.$$
- *Denoting by $s \mapsto \Gamma(s)$ the border $\Gamma_f(x_0)$ of $E_f(x_0)$, the oscillation exponent is $\beta_f(x_0) = -\Gamma'^{+}(h_f(x_0) - 1$ (where $g'^{+}(z)$ stands for the right derivative at z).*

So, the 2-microlocal domain provides one with a geometrical description of the pointwise regularity, that contains and generalizes all previous information, and has the enormous advantage to be immediately applicable to tempered distributions.

In addition, the property (11) makes it quite easy, once the domain $E_f(x_0)$ is known, to guess what will be the pointwise regularity of the derivatives or the primitives of f (at least, theoretically). This trick is used in several papers.

Having in mind the initial purpose of detecting and modeling local oscillations, it appears that the 2-microlocal spaces $C^{s,s'}_{x_0}$ are particularly well adapted. Indeed, a first step toward the decomposition of functions in a sum of a regular part and an oscillating part is provided by the following result from [39].

Theorem 3. *Assume either that $s + s' > 0$ and $s' < d$, or that $s' \geq n$ and $s + d > 0$. A distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ locally belongs to $C^{s,s'}_{x_0}$ if and only if locally around x_0,*
$$f(x) = g(x) + |x - x_0|^{-s'} h(x),$$
where :

- *$g \in C^{s,N}_{x_0}$ for every $N \in \mathbb{N}$,*
- *$h \in C^{s+s'}(\mathbb{R}^d)$ and $|h(x)| \leq C|x - x_0|^{s+s'}$.*

Hence, a function $f \in C^{s,s'}_{x_0}$ can be written the sum of a regular term and a singular term at x_0. These local decompositions can be pushed further, to effectively catch local oscillations, using the following definitions of (α, β)-chirps.

5. Decomposition of chirps and fast local oscillations of functions

The initial motivation for the description of local oscillations, essentially to see whether, around some points x_0, a function can effectively be compared to a chirp $|x - x_0|^\alpha \sin(|x - x_0|^{-\beta})$. The following definitions sets up a mathematical frame to these behaviors [32].

Definition 8. A real function h is indefinitely oscillating when for every $n \geq 1$, there exists an n-th primitive of h which is bounded at infinity.

Indefinitely oscillating functions are a first natural generalization of sine and cosine functions.

Definition 9. Let $\alpha \geq 0$, $\beta > 0$ and $r > 0$. A real function $f : \mathbb{R} \to \mathbb{R}$ is an (α, β)-chirp with regularity r at $x_0 \in \mathbb{R}$ when there exist two indefinitely oscillating functions v_+ and v_- such that, locally around x_0,

$$f(x) = (x - x_0)^\alpha v_+(|x - x_0|^{-\beta}) \text{ when } x \geq x_0$$
$$f(x) = (x_0 - x)^\alpha v_-(|x - x_0|^{-\beta}) \text{ when } x < x_0.$$

The main result, i.e. the characterization of (α, β)-chirps by means of 2-microlocal spaces, is the following:

Theorem 4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a locally integrable function. Let $x_0 \in \mathbb{R}$. The following statements are equivalent:*

(i) *There exists $\eta > 0$ such that on $[x_0 - \eta, x_0 + \eta]$, $f(x) = r(x) + g_{\alpha,\beta}(x)$, where $r \in C^\infty(\mathbb{R})$ and $g_{\alpha,\beta}$ is an (α, β)-chirp associated with indefinitely oscillating functions $v_+, v_- \in C^t([\eta^{-\beta}, +\infty))$.*

(ii) *f belongs locally to all 2-microlocal spaces $C^{s,s'}_{x_0}$ with indices s, s' satisfying $s + s' \geq t$ and $\alpha \geq (\beta + 1)s + \beta s'$.*

In Theorem 4.2 of [32], a characterization of (α, β)-chirps in terms of wavelet transform and wavelet coefficients is also proved. In all cases, it appears that the collection of 2-microlocal spaces plays a central role, and is well adapted to the description of local fluctuations.

This first characterization is an important step, but the precision can be enhanced using the notion of trigonometric chirps.

Definition 10. Let $\alpha \geq 0$, $\beta > 0$ and $r > 0$. A real function f as a trigonometric (α, β)-chirp with regularity r at $x_0 \in \mathbb{R}$ when for every integer $q \geq 1$, there exist two indefinitely oscillating function R^+_q, R^-_q, as well as g^+_0, g^-_0 ,g^+_1, g^-_1,... g^+_{q-1}, g^-_{q-1} $2q$ 2π-periodic functions with zero mean belonging to $C^{r+j}(\mathbb{R})$ such that, locally around x_0,

(12)

$$\text{when } x \geq x_0, \quad f(x) = (x - x_0)^\alpha (g^+_0(|x - x_0|^{-\beta}) + (x - x_0)^\beta (g^+_1(|x - x_0|^{-\beta}) + \cdots$$

$$+ (x - x_0)^{(q-1)\beta}(g^+_{q-1}(|x - x_0|^{-\beta}) + R^+_{q-1}(x), \tag{13}$$

and the same holds when $x \leq x_0$ by replacing the indices $+$ by $-$.

S. Jaffard and Y. Meyer found a wavelet characterization of trigonometric (α, β)-chirps (see Theorem 5.1 in [32]). We do not reproduce it here, since it is not so comfortable to write. Let us just say that around an trigonometric (α, β)-chirp x_0, the continuous wavelet transform $W_f(a, b)$ has an expansion similar to (12). Although this characterization is not easy to manipulate, it is doable and it allows them to analyze precisely the local behavior of the Riemann Fourier series (1). In particular,

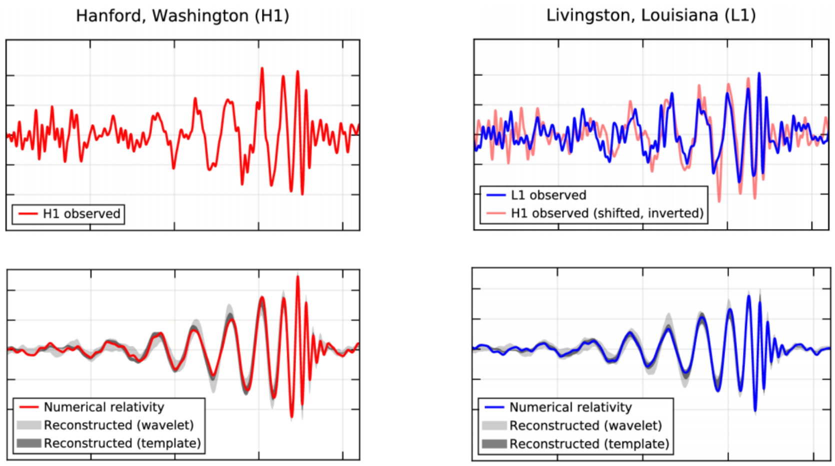

FIGURE 4. **Top:** Gravitational waves measured by LIGO and Livingston detectors on September 14th, 2015. Image credits from http ://dx.doi.org/10.7935/K5MW2F23, see [1] for details. **Bottom:** synthetic signal predicted from theory.

Theorem 5. *Let $x_0 = \frac{2p+1}{2q+1}$ be a rational number with odd numerator and denominator. Then around x_0, R is the sum of a C^∞ function and a trigonometric $(3/2, 1)$-chirp with regularity 1/2.*

It is remarkable that the periodic functions appearing in (12) for these trigonometric $(3/2, 1)$-chirps are essentially the Riemann series R itself, confirming the intuition that R possesses self-similar properties.

Such a theorem is the final outcome of more than a century of research around the Riemann series, after the works of Hardy, Gerver, Duistermaat [21], and many others. It was known for a while that R was differentiable only at those rational numbers that can be written $\frac{2p+1}{2q+1}$, but it was only suspected that oscillations were occurring around such points (cancellations effects explaining the exceptional large values of regularity locally).

The story was completed by the fine study of other remarkable points for the Riemann's series, the quadratic irrationals, which are proved to be logarithmic chirps of regularity 1/2, see [32] for details.

Other chirp decompositions have been studied in [41] for instance, and the detection and description of chirps is still an active domain of research in Mathematics, Physics and signal processing.

Remark 2. To be complete, let us precise that the chirps that are to be detected in gravitational waves for instance are not infinitely oscillating, but are locally "accelerating" as shown by Figure 5. The tools developed on the mathematical side prove to be useful to detect such behavior.

6. Applications to multifractals

The presence (or not) of oscillations is key in many situations and phenomena [40]. Let us conclude this article by explaining the role of oscillations in the multifractal formalism of functions.

Multifractals were introduced in Physics, in order to find an interpretation for experimental observations related to Kolmogorov's theory in fully developed turbulence. The fundamental work of Kolmogorov in the 1940's emphasized the role of the so-called scaling function, defined as follows: Let $v(x)$ be the velocity of a fluid in a bounded domain Ω, and consider

$$S_v(p,\ell) = \int_\Omega |v(x+\ell) - v(x)|^p dx.$$

Experiences show that, p being fixed,

$$S_v(p,\ell) \sim |\ell|^{\zeta_v(p)}$$

for some exponent $\zeta_v(p)$, when ℓ goes to zero. The mapping $p \mapsto \zeta_v(p)$ has been called the scaling function of the velocity $v(x)$. In the early 1980's, it has been shown on real data that $\zeta_v(p)$ was a strictly concave function, not anticipated before (for instance, in his K41 model, Kolmogorov predicted a linear shape for ζ_v). The main problem was then to understand this specific shape. In their article [22], Frisch and Parisi proposed an explanation of the concavity of $\zeta_v(p)$ using the fluctuations of the local regularity of $v(x)$.

Assume that the velocity v has various types of singularities: there exist an interval $I = [H_1, H_2] \subset \mathbb{R}^+$ such that for every $H \in I$, $E_v(H) := \{x : h_v(x_0) = H\}$ is non-empty. Such a function is called *multifractal*. The associated mapping

$$d_v : H \geq 0 \mapsto \dim E_v(H),$$

where dim stands for the Hausdorff dimension, is called the singularity spectrum of v. So, a function is multifractal when its singularity spectrum is not reduced to a single point. This spectrum describes the geometric distribution properties of the singularities of v.

Based on heuristic computations, Frisch and Parisi proposed in [22] a formula, now known in its generalized form as the multifractal formalism, who relates explicitly the singularity spectrum d_v with the scaling function $\zeta(p)$ via a Legendre transform:

$$d_v(h) = \inf_{p\in\mathbb{R}} (\zeta_v(p) - hp + 3). \tag{14}$$

It is remarkable that, up to some modifications, such a formula holds for large classes of functions. In addition, similar quantities can be defined for measures, and a multifractal formalism can be defined as well, whose validity is a whole and active field of research.

For instance, self-similar functions do satisfy a multifractal formalism [29].

It is striking that typical (or "generic") functions (in the sense of Baire category) in Besov and Sobolev spaces also satisfy a multifractal formalism. In [32], Y. Meyer found the optimal upper bound for the singularity spectrum of all functions in Sobolev and Besov spaces: for instance, when $s - d/p > 0$, the following holds:

Theorem 6. *For every* $f \in B^{s,p}_\infty$,

$$d_f(h) \leq \begin{cases} -\infty & \text{if } h < s - d/p, \\ ph - ps + d & \text{if } s - d/p \leq h \leq s, \\ d & \text{if } h > s, \end{cases}$$

$d_f(h) = -\infty$ *meaning that the corresponding level set* $E_f(h)$ *is empty.*

Further, it is proved in [31, 33] that the inequality above is generically realized, in the sense that typical functions $f \in B^{s,p}_\infty$ satisfy $d_f(h-(s-d/p)) = ph-ps+d$ for every $h \in [s-d/p, s]$, and $E_f(h) = \emptyset$ if $h \notin [s-d/p, s]$. As already said, such a result asserts that multifractality and fluctuations of pointwise regularity are common phenomena. Jaffard also proved that the multifractal formalism fails for such functions, and that this failure is due to the presence of chirps for typical functions. The understanding of oscillatory behaviors is key in these results, and has been investigated in [45] for instance.

Let us mention that in [15, 10] it is proved that measures supported by compact sets are typically multifractal, and satisfy a multifractal formalism for measures. Also, more recently, in [8, 9] a general framework for finding function spaces in which multifractal formalisms generically hold is developed.

Finally, the validity of multifractal formalism is extremely important for applications to signal processing. Indeed, it is unlikely that pointwise regularity exponents and singularity spectra can be precisely estimated on real data signals, since they involve various limits, Hausdorff or packing dimensions and sometimes fractional integration. Fortunately, the formula (14), when it holds, makes the estimation of the singularity spectrum d_f possible, since ζ_f (or equivalent quantities based on wavelet coefficients, wavelet leaders, local means, ...) are numerically accessible by log-log regressions on signals. Moreover, stable algorithms [46, 47, 2] have been recently developed that provide robust results for the estimation of multifractal parameters of signals coming from many scientific fields. These parameters are mainly used as classification tools for data, and the stability of their estimations explains the large success of the multifractal approach in image and signal processing.

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